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Some Classification Results for Hyperbolic Equations $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$

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We provide a contact invariant characterization for equations of the form

$$u_{xy} + a(x, y, u)\, u_x + b(x, y, u)\, u_y + c(x, y, u) = 0,$$

$$u_{xy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y, u) = 0,$$

$$u_{xy} + c(x, y, u) = 0,$$

$$u_{xy} = 0.$$

We classify all equations of the form $u_{xy} + f(x, y, u, u_x, u_y) = 0$ for which the two Ovsiannikov's invariants are constants. These results include characterization of the Klein–Gordon equation $u_{xy} = u$, the Liouville equation $u_{xy} = e^u$, and the class of Euler–Poisson–Darboux equations. It is shown that the wave equation $u_{xy} = 0$, Liouville equation, and the linear equation $u_{xy} = 2u/(x + y)^2$ are the only variational equations Darboux integrable at level one. We also show that a hyperbolic Monge–Ampére equation Darboux integrable at level one is equivalent to an equation of type $u_{xy} + f(x, y, u, u_x, u_y) = 0$. We prove that the hyperbolic Fermi–Ulam–Pasta (FPU) equation $u_{yy} = \kappa(u_x)^2\, u_{xx}$ is contact equivalent to a linear equation of type $u_{xy} = c(x + y)\, u$ and we classify all FPU equations Darboux integrable at level one. We also apply our results to equations of type $u_{xy} = F(u, u_x)$ that describe pseudo-spherical surfaces. © 2000 Academic Press

Key Words: generalized Laplace invariants; exact integrability; Darboux method; method of Laplace; *f*-Gordon equations; linear second-order scalar hyperbolic equations in the plane.

1. INTRODUCTION

A classical result of Lie states that the second-order scalar hyperbolic partial differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1.1}$$

[1] Some computations in this paper were performed with the help of the Maple package for the calculus of variations developed by Cinnamon Hillyard [15]. The author also thanks Ian Anderson for many valuable comments and suggestions.

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is contact equivalent to the wave equation $u_{xy}=0$ if and only if there are two functionally independent first-order integrals for each characteristic system of Eq. (1.1). The purpose of this paper is to present a variety of theorems which, in the spirit of Lie's theorem, provide simple characterizations of a number of well-known hyperbolic equations. In Section 3 we give a contact invariant characterization of equations of the form

$$u_{xy}+a(x, y, u)\, u_x+b(x, y, u)\, u_y+c(x, y, u)=0, \tag{1.2a}$$

$$u_{xy}+a(x, y)\, u_x+b(x, y)\, u_y+c(x, y, u)=0, \tag{1.2b}$$

$$u_{xy}+c(x, y, u)=0, \tag{1.2c}$$

and

$$u_{xy}=0. \tag{1.2d}$$

In Section 4 we generalize the characterization result of Ovsiannikov [21, p. 112], for linear equations to the equations of the form

$$u_{xy}+f(x, y, u, u_x, u_y)=0. \tag{1.3}$$

In particular we obtain the characterization of Klein–Gordon equation,

$$u_{xy}=u,$$

Liouville equation

$$u_{xy}=e^u,$$

and the class of Euler–Poisson–Darboux (EPD) equations.

In Section 5 we deduce that every Monge–Ampère equation Darboux integrable at level one is equivalent to an equation of type (1.3). From this follows that Goursat–Vessiot classification of equations of type (1.3) Darboux integrable at level one [13, 30] is actually a complete classification of Monge–Ampère equations that are Darboux integrable at level one. Independently of the results of Goursat and Vessiot we classify all variational equations Darboux integrable at level one. Namely, we show that *there are three non-equivalent classes of such equations which can be represented by*

$$u_{xy}=0, \qquad u_{xy}=e^u, \qquad \text{and} \qquad u_{xy}=\frac{2u}{(x+y)^2}.$$

In Section 6, as an application, we show that *the hyperbolic Fermi–Ulam–Pasta* (FPU) *equation*

$$u_{yy}=\kappa(u_x)^2\, u_{xx}$$

is contact equivalent to an equation of type

$$u_{xy} = c(x+y)\,u,$$

and that every hyperbolic FPU equation Darboux integrable at level one is contact equivalent to either the wave equation or the linear equation $u_{xy} = 2u/(x+y)^2$. This result refines the classification result for FPU equations Darboux integrable at level one obtained in [3]. We also obtain some new results on characterization of FPU equations Darboux integrable at higher levels, which are improvements over the results in [8, 14].

Next we find that equations of type

$$u_{xy} = F(u, u_x)$$

which describe η-pseudo-spherical surfaces are equivalent to one of the six equations

$$u_{xy} = \sin u, \qquad u_{xy} = u, \qquad u_{xy} = 0, \qquad u_{xy} = e^u,$$
$$u_{xy} = e^u u_x, \qquad \text{or} \qquad u_{xy} = e^u \sqrt{u_x^2 - 1}.$$

The last four equations are Darboux integrable at level one.

Finally we apply our results to the equation $u_{xy} + uu_{xx} + f(u_x) = 0$, studied by Calogero [5], to show that this equation is contact equivalent to a linear equation which is integrable by the method of Laplace at level one.

The methods of proof of our results are elementary and based upon the explicit formulas for the generalized Laplace invariants. Generalized Laplace invariants for hyperbolic equation (1.1) are two (possibly infinite) sequences $H_0, H_1, H_2, \ldots$, and $K_0, K_1, K_2, \ldots$, of relative contact invariants introduced in [1, 2]. The primary motivation behind the generalization of the classical Laplace invariants was to establish an analogy for non-linear equations of the following well-known characterization result for linear equations: A linear equation

$$u_{xy} + a(x, y)\,u_x + b(x, y)\,u_y + c(x, y)\,u = 0 \tag{1.4}$$

is integrable by the method of Darboux, if and only if the two sequences of the classical Laplace invariants are finite (see, for example, [9, 12] for details). In [1, 2], it is proved that if the equation (1.1) is Darboux integrable, then the two sequences of generalized Laplace invariants are finite. Later Sokolov and Zhiber [24] showed that this condition is sufficient for Darboux integrability of the equations of the form (1.3). Finally, Anderson and the author [18] showed that if the two sequences of generalized Laplace invariants are finite, then a hyperbolic equation

(1.1) is Darboux integrable. Combining the results of [1, 2, 18], we obtain a full generalization of the classical result: *A second-order scalar hyperbolic partial differential equation in the plane* (1.1) *is integrable by the method of Darboux if and only if the two sequences of generalized Laplace invariants are finite*.

In the turn of the century Darboux integrability attracted attention of giants such as Lie [19], Goursat [12, 13], and Cartan [6], and in the late 1930s and early 1940s of this century it was studied in remarkable papers of Vessiot [29, 30]. Recently it received a revived interest in a number of papers and books [1–3, 8, 10, 14, 20, 26–28] to mention but a few.

We now mention two characterization results from [17] that are not included in this paper.

A hyperbolic equation (1.1) *admits a complete intermediate integral if and only if at least one of the first two generalized Laplace invariants H_0 or K_0 vanishes.*

A hyperbolic Monge–Ampère equation (1.1) *admits a general intermediate integral if and only if at least one of the first two generalized Laplace invariants H_0 or K_0 vanishes.*

Finally, we would like to note, that we found new invariants whose vanishing is a necessary and sufficient condition for the hyperbolic equation (1.1) to have at least one invariant function for a given characteristic system. Characterization of equations of the form (1.3) then follows immediately (see Theorem 3.1 below). These and other results will be presented elsewhere.

2. PRELIMINARIES

We now briefly recall an important generalization of the classical Laplace invariants introduced in [1, 2]. Consider a second-order equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{2.1}$$

satisfying

$$\left(\frac{\partial F}{\partial u_{xx}}, \frac{\partial F}{\partial u_{xy}}, \frac{\partial F}{\partial u_{yy}}\right) \neq 0. \tag{2.2}$$

Equation (2.1) is called *hyperbolic* if the *characteristic equation*

$$\frac{\partial F}{\partial r}\lambda^2 - \frac{\partial F}{\partial s}\lambda\mu + \frac{\partial F}{\partial t}\mu^2 = 0, \tag{2.3}$$

defined at the points (2.1), has a pair of non-proportional second-order real roots $(\mu, \lambda)=(m_x, m_y)$ and $(\lambda, \mu)=(n_x, n_y)$ which define the *characteristic* (*total*) *vector fields*

$$X=m_x D_x+m_y D_y \qquad \text{and} \qquad Y=n_x D_x+n_y D_y. \tag{2.4}$$

Here D_x and D_y denote the total derivative operators for (2.1).

EXAMPLE 2.1. For the equation

$$u_{xx}+f(x, y, u, u_x, u_y, u_{xy}, u_{yy})=0,$$

the total derivative operators D_x and D_y are given by

$$D_y=\frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{xy}\frac{\partial}{\partial u_x}+u_{yy}\frac{\partial}{\partial u_y}+u_{xyy}\frac{\partial}{\partial u_{xy}}+u_{yyy}\frac{\partial}{\partial u_{yy}}+\cdots,$$

and

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}-f\frac{\partial}{\partial u_x}+u_{xy}\frac{\partial}{\partial u_y}-D_y f\frac{\partial}{\partial u_{xy}}+u_{xyy}\frac{\partial}{\partial u_{yy}}+\cdots$$

EXAMPLE 2.2. For the equation

$$u_{xy}+f(x, y, u, u_x, u_y)=0,$$

the total derivative operators D_x and D_y are

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}-f\frac{\partial}{\partial u_y}+u_{xxx}\frac{\partial}{\partial u_{xx}}-D_y f\frac{\partial}{\partial u_{yy}}+\cdots$$

and

$$D_y=\frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}-f\frac{\partial}{\partial u_x}+u_{yy}\frac{\partial}{\partial u_y}-D_x f\frac{\partial}{\partial u_{xx}}+u_{yyy}\frac{\partial}{\partial u_{yy}}+\cdots$$

We write the Lie bracket of the characteristic vector fields as

$$[X, Y]=PX+QY, \tag{2.5}$$

for some functions P, Q. For Eq. (2.1) we define a linear differential operator

$$\begin{aligned} L_F(\varphi)=&\frac{\partial F}{\partial u_{xx}}D_x\,D_x(\varphi)+\frac{\partial F}{\partial u_{xy}}D_x\,D_y(\varphi)\\ &+\frac{\partial F}{\partial u_{yy}}D_y\,D_y(\varphi)+\frac{\partial F}{\partial u_x}D_x(\varphi)+\frac{\partial F}{\partial u_y}D_y(\varphi)+\frac{\partial F}{\partial u}\varphi, \end{aligned} \tag{2.6}$$

which, in terms of the of the characteristic vector fields, can be written as

$$XY(\varphi)+AX(\varphi)+BY(\varphi)+C\varphi=0$$

or (2.7)

$$YX(\varphi)+DX(\varphi)+EY(\varphi)+G\varphi=0.$$

Explicit formulas for the coefficients A, B, ..., G and P, Q, are not difficult to obtain and can be found in [2]. In [1, 2] two sequences of relative contact invariants for equation (2.1) were introduced. The first two invariants are defined in terms of the coefficients of the linearizations (2.7) by

$$H_0=X(A)+AB-C \qquad \text{and} \qquad K_0=Y(E)+DE-G.$$

The higher level invariants H_i and K_j are defined recursively, provided $H_i\neq 0$ and $K_j\neq 0$, as

$$H_{i+1}=2H_i-H_{i-1}-XY(\ln|H_i|)+QY(\ln|H_i|)+Y(Q)+2PQ-X(P)$$

and

$$K_{j+1}=2K_j-K_{j-1}-YX(\ln|K_j|)-PX(\ln|K_j|)+Y(Q)+2PQ-X(P),$$

where $i, j\geqslant 0$ and where $H_{-1}=K_0$ and $K_{-1}=H_0$ [17]. The invariants $H_0, H_1, H_2, \ldots,$ and $K_0, K_1, K_2, \ldots,$ are called *generalized Laplace invariants.*

Recall that Eq. (2.1) is (*contact*) *equivalent* to the equation

$$\bar{F}(\bar{x}, \bar{y}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{y}}, \bar{u}_{\bar{y}\bar{y}})=0. \tag{2.8}$$

if there is an invertible transformation $(x, y, u, u_x, u_y)\rightarrow(\bar{x}, \bar{y}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}})$, such that solutions of (2.1) are mapped to solutions of (2.8), i.e., omitting the pullback notation, we have

$$d\bar{u}-\bar{u}_{\bar{x}}\,d\bar{x}-\bar{u}_{\bar{y}}\,d\bar{y}=\lambda(du-u_x\,dx-u_y\,dy)$$

for some function λ. Such transformations are called (*invertible*) *contact transformations*. The transformation rules for $\bar{u}_{\bar{x}\bar{x}}$, $\bar{u}_{\bar{x}\bar{y}}$, and $\bar{u}_{\bar{y}\bar{y}}$, are determined from the fact that the pullback of the two one-forms

$$d\bar{u}_{\bar{x}}-\bar{u}_{\bar{x}\bar{x}}\,d\bar{x}-\bar{u}_{\bar{x}\bar{y}}\,d\bar{y} \qquad \text{and} \qquad d\bar{u}_{\bar{y}}-\bar{u}_{\bar{x}\bar{y}}\,d\bar{x}-\bar{u}_{\bar{y}\bar{y}}\,d\bar{y}$$

is in the span of

$$du-u_x\,dx-u_y\,dy, \qquad du_x-u_{xx}\,dx-u_{xy}\,dy,$$

and

$$du_y - u_{xy}\,dx - u_{yy}\,dy.$$

EXAMPLE 2.3. One can readily verify that the hyperbolic equation

$$u_{xx}u_{yy} - u_{xy}^2 + 1 = 0$$

is contact equivalent to the wave equation $\bar{u}_{\bar{x}\bar{y}} = 0$ and the contact transformation is given by

$$\bar{x} = x + u_y, \qquad \bar{y} = y + u_x, \qquad \bar{u} = u + \tfrac{1}{2}(u_x u_y - x u_x - y u_y - xy),$$

$$\bar{u}_{\bar{x}} = \tfrac{1}{2}(u_x - y), \qquad \text{and} \qquad \bar{u}_{\bar{y}} = \tfrac{1}{2}(u_y - x).$$

We remark that H_j/H_i and K_j/H_i are absolute contact invariants [2], provided $H_i \neq 0$. More specifically, for an invertible contact transformation $(x, y, u, u_x, u_y) \to (\bar{x}, \bar{y}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}})$, there exists a second-order function $m = m(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ which does not vanish on solutions of (2.1) such that $\bar{H}_i = mH_i$ and $\bar{K}_j = mK_j$, for all $0 \leqslant i, j$.

Later in the text we will make use of the following theorem which we state without proof. This theorem follows immediately from the main result (Theorem 5.3) of [18] and from the results of [17, Chap. 10]. To formulate the result we need to introduce several definitions.

We say that a function $I = \varphi(x, y, u, u_x, u_y, \ldots)$ which is not constant along all solutions of (2.1) is *X invariant* if $X(I) = 0$ for all solutions of Eq. (2.1). Similarly, we say that a function $J = \phi(x, y, u, u_x, u_y, \ldots)$ which is not constant along all solutions of (2.1) is *Y invariant* if $Y(J) = 0$ for all solutions of Eq. (2.1). Let **R** denotes the set of real numbers. We say that two functions $I = \varphi(x, y, u, u_x, u_y, \ldots)$ and $J = \phi(x, y, u, u_x, u_y, \ldots)$ are *functionally independent* (*for Eq.* (2.1)) if for every open subset $U \subseteq \mathbf{R}$ and every function $G: U \to \mathbf{R}$ such that $G(I, J) = 0$, for all solutions $u: U \to \mathbf{R}$ of Eq. (2.1), follows that G is identically zero on U.

An important class of equations invariant under contact transformations is the class of *Monge–Ampère equations*, i.e., equations of the form

$$E(u_{xx}u_{yy} - u_{xy}^2) + Au_{xx} + 2Bu_{xy} + Cu_{yy} + D = 0, \tag{2.9}$$

where the functions A, B, C, D, and E, depend only on x, y, u, u_x, and u_y.

THEOREM 2.4. *Assume that for a hyperbolic equation* (2.1) *one of the following conditions is satisfied*:

(i) $H_0 = K_0 = 0$.

(ii) *Equation* (2.1) *is Monge–Ampère and* $H_0 = 0$, $K_0 \neq 0$, *and* $K_1 = 0$.

(iii) *Equation* (2.1) *is Monge–Ampère and* $H_0 \neq 0$, $K_0 = 0$, *and* $H_1 = 0$.

(iv) *Equation* (2.1) *is Monge–Ampère and* $H_0 \neq 0$, $K_0 \neq 0$, $H_1 = 0$, *and* $K_1 = 0$.

Then there is a pair of X invariant functionally independent functions $I_1 = \varphi_1(x, y, u, u_x, u_y)$ *and* $I_2 = \varphi_2(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$, *and there is a pair of Y invariant functionally independent functions* $J_1 = \phi_1(x, y, u, u_x, u_y)$ *and* $J_2 = \phi_2(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$.

EXAMPLE 2.5. Consider the hyperbolic equation $u_{xy} = u_x e^u$ with the characteristic vector fields $X = D_x$, $Y = D_y$. We easily compute $H_0 = 0$, $K_0 \neq 0$, and $K_1 = 0$. The functions $I_1 = y$ and $I_2 = u_y - e^u$ are X invariant and the functions $J_1 = x$ and $J_2 = u_{xx}/u_x - u_x$ are Y invariant.

3. CHARACTERIZATION OF CERTAIN SUBCLASSES OF *f*-GORDON EQUATIONS

For the rest of this paper we adopt the Monge notation: $u_x = p$, $u_y = q$, $u_{xx} = r$, $u_{xy} = s$, and $u_{yy} = t$. Consider the hyperbolic equation

$$F(x, y, u, p, q, r, s, t) = 0, \tag{3.1}$$

satisfying the non-degeneracy condition (2.2), with characteristic vector fields X and Y. An *f-Gordon equation* is an equation of the form

$$s + f(x, y, u, p, q) = 0. \tag{3.2}$$

The following theorem is a fundamental classification result for *f*-Gordon equations:

THEOREM 3.1. *The hyperbolic equation* (3.1) *is contact equivalent to an f-Gordon equation if and only if there are two first-order functions I and J, i.e., functions depending on x, y, u, and first derivatives of u, such that I is X invariant and J is Y invariant.*

From Theorem 3.1 follows that if any of the conditions (i), ..., (iv) of Theorem 2.4 is satisfied for Eq. (3.1), then the Eq. (3.1) if *f*-Gordon.

For an *f*-Gordon equation (3.2) we choose characteristic vector fields $X = D_x$ and $Y = D_y$ and so the first generalized Laplace invariants are

$$H_0 = D_x\left(\frac{\partial f}{\partial p}\right) + \frac{\partial f}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial f}{\partial u} \qquad \text{and} \qquad K_0 = D_y\left(\frac{\partial f}{\partial q}\right) + \frac{\partial f}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial f}{\partial u}.$$

If $H_i \neq 0$ and $K_j \neq 0$, we have

$$H_{i+1} = 2H_i - H_{i-1} - D_x\, D_y(\ln |H_i|)$$

and (3.3)

$$K_{j+1} = 2K_j - K_{j-1} - D_x\, D_y(\ln |K_j|),$$

where $i,\ j \geqslant 0$ and where $H_{-1} = K_0$ and $K_{-1} = H_0$.

Anderson and Kamran [1, Proposition 14.4] showed that the condition $H_0 = K_0 = 0$ is necessary and sufficient for an f-Gordon equation (3.2) to be contact equivalent to the wave equation. Their proof relies on the characterization the contact orbit of the wave equation by Gardner and Kamran [10]. Theorem 2.4 is the key to the generalization of this result.

THEOREM 3.2. *A hyperbolic equation* (3.1) *is contact equivalent to the wave equation* $s = 0$ *if and only if* $H_0 = K_0 = 0$.

Proof. If the hyperbolic equation (3.1) satisfies $H_0 = K_0 = 0$, then by Theorem 2.4, it has X and Y invariant first-order functions and so by Theorem 3.1 the equation is equivalent to an f-Gordon equation (3.2). We have

$$H_0 = \frac{\partial f}{\partial p\, \partial p} r + \frac{\partial f}{\partial x\, \partial p} + \frac{\partial f}{\partial u\, \partial p} p - \frac{\partial f}{\partial p\, \partial q} f + \frac{\partial f}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial f}{\partial u}, \tag{3.4}$$

$$K_0 = \frac{\partial f}{\partial q\, \partial q} t + \frac{\partial f}{\partial y\, \partial q} + \frac{\partial f}{\partial u\, \partial q} q - \frac{\partial f}{\partial p\, \partial q} f + \frac{\partial f}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial f}{\partial u}. \tag{3.5}$$

From $H_0 = K_0 = 0$ follows $\partial f/\partial p\, \partial p = 0$ and $\partial f/\partial q\, \partial q = 0$, that is, Eq. (3.1) is equivalent to an equation

$$s + g(x,\ y,\ u)\, pq + a(x,\ y,\ u)\, p + b(x,\ y,\ u)\, q + c(x,\ y,\ u) = 0. \tag{3.6}$$

Consider the point transformation $\bar{x} = x$, $\bar{y} = y$, and $\bar{u} = \bar{u}(x,\ y,\ u)$, where $\bar{u}$ is determined by the equation $\partial \bar{u}/\partial u = e^G$ and where G satisfies $\partial G/\partial u = g$. Eq. (3.6) is transformed into an equation of the form

$$s + a(x, y, u) p + b(x, y, u)\, q + c(x, y, u) = 0. \tag{3.7}$$

From $H_0 = K_0 = 0$ we conclude $\partial a/\partial u = \partial b/\partial u = 0$ and $\partial a/\partial x = \partial b/\partial y$. These conditions imply the existence of the function $\varphi = \varphi(x, y)$, such that $\partial \varphi/\partial y = a$ and $\partial \varphi/\partial x = b$. Under the point transformation $\bar{x} = x$, $\bar{y} = y$, and $\bar{u} = e^{\varphi} u$, Eq. (3.7) is transformed into an equation of the form

$$s + c(x, y, u) = 0. \tag{3.8}$$

Since $\partial c/\partial u = -H_0 = 0$, then $c = c(x, y)$. The point transformation $\bar{x} = x$, $\bar{y} = y$, and $\bar{u} = u + C$, where $C = C(x, y)$ is a function satisfying the differential equation $\partial C/\partial x\, \partial y = c$, carries (3.8) into the wave equation. ∎

THEOREM 3.3. *Let Eq.* (3.1) *be contact equivalent to an f-Gordon equation with* $H_0 \neq 0$ *and* $K_0 \neq 0$. *Then* (3.1) *is contact equivalent to an equation of form*

$$s + a(x, y, u)\, p + b(x, y, u)\, q + c(x, y, u) = 0 \tag{3.9}$$

if and only if K_0/H_0 *is of order* 1, *i.e., depends on* x, y, u, p, *and* q *only.*

Proof. An easy computation shows that K_0/H_0 for an equation of form (3.9) is a function of order one. We now prove the converse. Consider the f-Gordon equation (3.2). We arrive at H_0 and K_0 given by (3.4) and (3.5). Since H_0/K_0 is of order one, we deduce $\partial f/\partial p\, \partial p = 0$ and $\partial f/\partial q\, \partial q = 0$, and so (3.2) is of the form (3.6). Consider the point transformation $\bar{x} = x$, $\bar{y} = y$, and $\bar{u} = \bar{u}(x, y, u)$, where $\bar{u}$ is determined by $\partial \bar{u}/\partial u = \exp(\int g\, du)$. This transformation transforms Eq. (3.6) into the equation of the form of (3.9). ∎

THEOREM 3.4. *Let Eq.* (3.1) *be contact equivalent to an f-Gordon equation. If* $H_0 \neq 0$, $K_0 \neq 0$, $X(K_0/H_0) \neq 0$, *and* $Y(H_0/K_0) \neq 0$, *then* (3.1) *is contact equivalent to an equation of form*

$$s + a(x, y)\, p + b(x, y)\, q + c(x, y, u) = 0, \tag{3.10}$$

if and only if the following conditions are satisfied.

(i) H_0/K_0 *is of order one and*

(ii) $L = \dfrac{1}{H_0} X\left(\dfrac{K_0}{H_0}\right) Y\left(\dfrac{H_0}{K_0}\right)$ *is of order one.*

Proof. We remark that L is an absolute contact invariant. An easy computation shows that conditions (i) and (ii) are satisfied for the equation of the form (3.10). Conversely, assume that (i) and (ii) are satisfied. By Theorem 3.3 we may assume that the equation is given by (3.9). We now prove that $\partial a/\partial u = \partial b/\partial u = 0$. The first generalized Laplace invariants H_0 and K_0 are

$$H_0 = -\frac{\partial b}{\partial u} q + \frac{\partial a}{\partial x} - \frac{\partial c}{\partial u} + ab \qquad \text{and} \qquad K_0 = -\frac{\partial a}{\partial u} p + \frac{\partial b}{\partial y} - \frac{\partial c}{\partial u} + ab. \tag{3.11}$$

An easy computation yields

$$X\left(\frac{K_0}{H_0}\right)=D_x\left(\frac{K_0}{H_0}\right)=-\frac{r}{H_0}\frac{\partial a}{\partial u}+\alpha_1\neq 0, \tag{3.12}$$

$$Y\left(\frac{H_0}{K_0}\right)=D_y\left(\frac{H_0}{K_0}\right)=-\frac{t}{K_0}\frac{\partial b}{\partial u}+\alpha_2\neq 0, \tag{3.13}$$

where α_1, α_2 are functions of order 1. Hence

$$L=\frac{1}{H_0^2K_0}\frac{\partial a}{\partial u}\frac{\partial b}{\partial u}rt-\frac{\alpha_2}{H_0^2}\frac{\partial a}{\partial u}r-\frac{\alpha_1}{H_0K_0}\frac{\partial b}{\partial u}t+\alpha_1\alpha_2. \tag{3.14}$$

If $\alpha_1=\alpha_2=0$, then from the last equation follows $\partial a/\partial u=0$ or $\partial b/\partial u=0$, which yields a contradiction with (3.12) or (3.13). Consequently, either α_1 or α_2 does not vanish. Say $\alpha_1\neq 0$. Then from (3.14) follows $\partial b/\partial u=0$. By (3.13) $\alpha_2\neq 0$, and so from (3.14) we deduce $\partial a/\partial u=0$. If $\alpha_2\neq 0$, then a similar argument shows that $\partial a/\partial u=0$ and $\partial b/\partial u=0$. ▮

THEOREM 3.5. *Let equation Eq.* (3.1) *be contact equivalent to an f-Gordon equation. Then* (3.1) *is contact equivalent to an equation of form*

$$s+c(x, y, u)=0, \tag{3.15}$$

if and only if $H_0=K_0$.

Proof. Assume that the equation is of the form (3.15). Then $H_0=K_0$ follows by an easy computation. We now prove the converse. Consider the f-Gordon equation (3.2) with $H_0=K_0$. If $H_0=K_0=0$, then by Theorem 3.2 the equation is contact equivalent to the wave equation and the statement follows. Assume $H_0\neq 0$. By Theorem 3.3, Eq. (3.1) is given by (3.9) and so H_0, and K_0 are as in (3.11), from which follows $\partial a/\partial u=\partial b/\partial u=0$ and $\partial a/\partial x=\partial b/\partial y$. Thus there is a function $\varphi=\varphi(x, y)$ satisfying $\partial\varphi/\partial y=a$ and $\partial\varphi/\partial x=b$ and so under the point transformation $\bar{x}=x$, $\bar{y}=y$, and $\bar{u}=e^{\varphi}u$, Eq. (3.9) becomes of the form (3.15). ▮

THEOREM 3.6. *The f-Gordon equation* (3.2) *is contact equivalent to an Euler–Lagrange equation of a first-order Lagrangian if and only if* $H_0=K_0$.

Proof. Consider the first-order Lagrangian $L(x, y, u, p, q)$. It is easy to check that that the Euler–Lagrange equation

$$\frac{\partial L}{\partial u}-D_x\frac{\partial L}{\partial p}-D_y\frac{\partial L}{\partial q}=0$$

satisfies $H_0 = K_0$. Conversly, by Theorem 3.5, Eq. (3.2) satisfying $H_0 = K_0$ is equivalent to an equation $s + c(x, y, u) = 0$, which is variational (consider the Lagrangian $L = (-1/2)\, pq + \int c\, du$). ∎

From Theorem 3.5 and Theorem 3.6 immediately follows

COROLLARY 3.7. *An f-Gordon equation is contact equivalent to an Euler–Lagrange equation of a first-order Lagrangian if and only if it is contact equivalent to some equation of the form* $s = c(x, y, u)$.

4. CHARACTERIZATION OF *f*-GORDON EQUATIONS WITH CONSTANT OVSIANNIKOV'S INVARIANTS

Following Ibragimov [16, p. 45] we call the functions $\sigma_0 = K_0/H_0$ and $\sigma_1 = (2H_0 - K_0 - H_1)/H_0$ *Ovsiannikov's invariants*. We will now turn to the case when these two functions are constants, i.e., the case when K_0/H_0 and H_1/H_0 are constants. We start with the following lemma.

LEMMA 4.1. *Let Eq.* (3.1) *be contact equivalent to an f-Gordon equation and* $k \neq 0, 1$ *be a constant. Then* $K_0 = kH_0$, *if and only if* (3.1) *is contact equivalent to a linear equation of the form*

$$s + a(x, y)\, p + b(x, y)\, q + c(x, y)\, u = 0, \tag{4.1}$$

where

$$c = \frac{1}{k-1}\left[k\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right] + ab. \tag{4.2}$$

Proof. If Eq. (3.1) satisfies (4.1) and (4.2), then by a simple computation $K_0 = kH_0$. Now assume $K_0 = kH_0$. If $H_0 = K_0 = 0$, then we have the wave equation. Let $K_0/H_0 = k$. By Theorem 3.3, Eq. (3.1) is contact equivalent to the equation of the form (3.9) and so H_0 and K_0 are given by (3.11). We deduce that

$$\frac{\partial a}{\partial u} = \frac{\partial b}{\partial u} = 0, \qquad \text{and} \qquad k\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} = (k-1)\left(\frac{\partial c}{\partial u} - ab\right).$$

Differentiating the last equation with respect to u, we obtain $\partial c/\partial u\, \partial u = 0$, and so the equation can be written in the form

$$s + a(x, y)\, p + b(x, y)\, q + g(x, y)\, u + h(x, y) = 0. \tag{4.3}$$

Let $\varphi=\varphi(x, y)$ be a particular solution to (4.3). Then the transformation $\bar{x}=x$, $\bar{y}=y$, and $\bar{u}=u-\varphi(x, y)$ carries Eq. (4.3) into a linear equation of the form (4.1). The condition (4.2) follows immediately from the condition $K_0=kH_0$. ∎

Lemma 4.1 and the characterization theorem of Ovsiannikov [21, p. 112], combine to establish the following result:

THEOREM 4.2. *Let Eq.* (3.1) *be contact equivalent to an f-Gordon equation and* $H_0\neq 0$.

(i) $\sigma_0\neq 0, 1$ *and* $\sigma_1\neq 0$ *are constants if and only if* (3.1) *is contact equivalent to the Euler–Poisson–Darboux* (*EPD*) *equation*

$$s-\frac{2}{\sigma_1(x+y)}\,p-\frac{2\sigma_0}{\sigma_1(x+y)}\,q+\frac{4\sigma_0}{\sigma_1^2(x+y)^2}\,u=0.$$

(ii) $\sigma_0\neq 0, 1$ *is a constant and* $\sigma_1=0$ *are if and only if* (3.1) *is contact equivalent to the linear equation*

$$s+xp+\sigma_0\,yq+\sigma_0\,xyu=0,$$

provided $c=0$.

We will now treat the case $H_0=K_0$ ($\sigma_0=1$), i.e., the case of variational equations.

THEOREM 4.3. *Let Eq.* (3.1) *be contact equivalent to an f-Gordon equation and let* $H_0=K_0\neq 0$.

(i) $H_1/H_0=0$ *if and only if* (3.1) *is contact equivalent either to Liouville equation* $s=e^u$ *or to EPD equation* $s=2u/(x+y)^2$.

(ii) $H_1/H_0=1$ *if and only if Eq.* (3.1) *is contact equivalent to Klein–Gordon equation* $s=u$.

(iii) $H_1/H_0=k$, *where* k *is a real number and* $k\neq 0, 1$, *if and only if Eq.* (3.1) *is contact equivalent to EPD equation*

$$s=\frac{2u}{(1-k)(x+y)^2}. \tag{4.4}$$

Proof. One can easily see that the invarince conditions are satisfied by each of the equations. Conversely, let $H_0=K_0\neq 0$ and $H_1/H_0=k$ be satisfied for an f-Gordon equation (3.2). From Theorem 3.5 we conclude that the equation is given by $s+c(x, y, u)=0$ and so $H_0=-\partial c/\partial u$. Using the recursion formulas (3.3) we conclude from the highest order coefficients of $H_1 = kH_0$ that

$$\frac{\partial c}{\partial u}\frac{\partial c}{\partial u\,\partial u\,\partial u} - \left(\frac{\partial c}{\partial u\,\partial u}\right)^2 = 0. \tag{4.5}$$

Integrating, we obtain

$$c = \pm\frac{1}{a}e^{au+b} + g, \qquad \text{where} \qquad a \neq 0, \tag{4.6}$$

or

$$c = \pm\, e^a u + b, \tag{4.7}$$

where a, b, g are arbitrary functions of x and y only. Note that $c = c(x, y)$ is also a solution to (4.5) but it yields $H_0 = K_0 = 0$, which is a contradiction to our assumption. First assume that c is given by (4.6). Then $H_0 = K_0 = \mp\, e^{au+b}$ and using (3.3) we arrive at

$$H_1 = -\frac{\partial a}{\partial y}p - \frac{\partial a}{\partial x}q - \frac{\partial a}{\partial x\,\partial y}u + ag - \frac{\partial b}{\partial x\,\partial y}.$$

From $H_1 = kH_0$ we obtain $\partial a/\partial x = 0$ and $\partial a/\partial y = 0$, and so a is a constant. We conclude that

$$ag - \frac{\partial b}{\partial x \partial y} = \mp\, ke^{au+b}.$$

Differentiating the last equation with respect to u yields $ka = 0$. If $k \neq 0$, then $a = 0$ which is a contradiction. If $k = 0$, then $g = (1/a)\,\partial b/\partial x\,\partial y$, and so the equation is equivalent to the equation

$$s \pm \frac{1}{a}e^{au+b} + \frac{1}{a}\frac{\partial b}{\partial x\,\partial y} = 0. \tag{4.8}$$

The point transformation $\bar{x} = \mp\, x$, $\bar{y} = y$, and $\bar{u} = au + b$ transforms (4.8) into $\bar{s} = e^{\bar{u}}$.

Next assume that the function c is given by (4.7). Thus, the equation is equivalent to

$$s \pm e^a u + b = 0, \tag{4.9}$$

and so $H_0 = {}^{-}_{+}\, e^a$ and $H_1 = {}^{-}_{+}\, e^a - \frac{\partial a}{\partial x\, \partial y}$. From $H_1 = kH_0$ we deduce that

$$\frac{\partial a}{\partial x\, \partial y} = {}^{+}_{-}\, (k-1)\, e^a. \tag{4.10}$$

If $k=1$, then $\partial a/\partial x\, \partial y = 0$ and so $a = \varphi(x) + \psi(y)$, where $\varphi(x)$ and $\psi(y)$ are arbitrary functions. Denote $F'(x) = {}^{+}_{-}\, e^{\varphi(x)}$ and $G'(y) = e^{\phi(y)}$. The point transformation $\bar{x} = F(x)$, $\bar{y} = G(y)$, and $\bar{u} = u$, transforms Eq. (4.9) into $\bar{s} + \bar{u} + \bar{b}(\bar{x}, \bar{y}) = 0$, for some function $\bar{b}$. The point transformation $x = -\bar{x}$, $y = \bar{y}$, and $u = \bar{u} - \xi(\bar{x}, \bar{y})$, where ξ satisfies the last equation, carries the equation into the Klein–Gordon equation $s = u$.

If $k \neq 1$, then integrating (4.10) we obtain

$$e^a = {}^{+}_{-}\, \frac{2\varphi'(x)\, \psi'(y)}{(k-1)(\varphi(x) + \psi(y))^2},$$

where φ and ψ are arbitrary functions. Thus, the point transformation $\bar{x} = \varphi(x)$, $\bar{y} = \psi(y)$, and $\bar{u} = u$ transforms (4.9) into the equation $\bar{s} + 2\bar{u}/((k-1)(\bar{x} + \bar{y})^2) + \bar{b}(\bar{x}, \bar{y}) = 0$, for some function $\bar{b}$. Finally, the point transformation $x = \bar{x}$, $y = \bar{y}$, and $u = \bar{u} - \phi(\bar{x}, \bar{y})$, where ϕ satisfies the last equation carries this equation into (4.4). ∎

It is obvious that we cannot distinguish between the Liouville equation and the equation $s = 2u/(x+y)^2$ in terms of the generalized Laplace invariants. But, the infinitesimal contact symmetry algebra for Liouville equation is

$$\mathbf{g}_1 = \left\{ \mathrm{pr}^2 v;\ v = -\psi(x)\, \frac{\partial}{\partial x} - \xi(y)\, \frac{\partial}{\partial y} + (\psi'(x) + \xi'(y))\, \frac{\partial}{\partial u} \right\}$$

and the infinitesimal contact symmetry algebra for the equation $s = 2u/(x+y)^2$ is

$$\mathbf{g}_2 = \left\{ \mathrm{pr}^2 v;\ v = -(ax+b)\, \frac{\partial}{\partial x} - (ay-b)\, \frac{\partial}{\partial y} \right.$$
$$\left. + \left(cu + \frac{\psi(x) + \xi(y)}{x+y} - \frac{\psi'(x) + \xi'(y)}{2} \right) \frac{\partial}{\partial u} \right\},$$

where $\mathrm{pr}^2 v$ is the second prolongation of the vector field v, and a, b, c are constants, and $\psi(x)$, $\xi(y)$ are arbitrary functions. Since $[\mathbf{g}_1, \mathbf{g}_1] = \mathbf{g}_1$ and $[\mathbf{g}_2, \mathbf{g}_2] \neq \mathbf{g}_2$, then $\mathbf{g}_1$ is not isomorfic to $\mathbf{g}_2$, and so the two equations are not equivalent.

5. CHARACTERIZATION OF MONGE-AMPÈRE EQUATIONS DARBOUX INTEGRABLE AT LEVEL ONE

DEFINITION 5.1. We say that the hyperbolic equation (3.1) is *Darboux integrable at level k* if there are two functionally independent X invariant functions I_1 and I_2 depending on x, y, u and derivatives of u up to order $k+1$ and two functionally independent Y invariant functions I_1 and I_2 depending on x, y, u and derivatives of u up to order $k+1$.

The essence of the method of Darboux lies in the observation that the Eq. (3.1) together with the two equations $I_2 = f(I_1)$ and $J_2 = g(J_1)$, where f and g are arbitrary functions, form an integrable system.

EXAMPLE 5.2. We consider Darboux integrable equation

$$u_{xy} + \frac{2\sqrt{u_x u_y}}{x+y} = 0, \tag{5.1}$$

studied by Goursat [12, p. 171, Example XI] (see also Vessiot [30]). The characteristic vector field $X = D_x$ has invariant functions $I_1 = y$ and $I_2 = u_{yy}/(2\sqrt{u_y}) + \sqrt{u_y}/(x+y)$ and the characteristic vector field $Y = D_y$ has invariant functions $J_1 = x$ and $J_2 = u_{xx}/(2\sqrt{u_x}) + \sqrt{u_x}/(x+y)$. The three equations

$$u_{xy} + \frac{2\sqrt{u_x u_y}}{x+y} = 0, \qquad \frac{u_{xx}}{2\sqrt{u_x}} + \frac{\sqrt{u_x}}{x+y} = f(x), \qquad \frac{u_{yy}}{2\sqrt{u_y}} + \frac{\sqrt{u_y}}{x+y} = g(y),$$

where $f(x)$ and $g(y)$ are arbitrary functions, give rise to the Pfaffian system generated by

$$\theta_1 = du - u_x\, dx - u_y\, dy,$$

$$\theta_2 = du_x + \left(\frac{2u_x}{x+y} - 2\sqrt{u_x}\, f(x)\right) dx + \frac{2\sqrt{u_x u_y}}{x+y}\, dy,$$

$$\theta_3 = du_y + \frac{2\sqrt{u_x u_y}}{x+y}\, dx + \left(\frac{2u_y}{x+y} - 2\sqrt{u_y}\, g(y)\right) dy,$$

with the independence condition $dx \wedge dy \neq 0$. It is easy to check that the system $\{\theta_1, \theta_2, \theta_3\}$ is integrable. Finding the integral surfaces to this system, i.e two-dimensional manifolds on which θ^1, θ^2, and θ^3 vanish, amounts to integrating Eq. (5.1). We conclude that

$$\sqrt{u_x} = F'(x) - \frac{F(x) + G(y)}{x+y}, \qquad \sqrt{u_y} = G'(y) - \frac{F(x) + G(y)}{x+y},$$

and

$$u = -\frac{(F(x)+G(y))^2}{x+y} + \int F'(x)^2\,dx + \int G'(y)^2\,dy, \tag{5.2}$$

where $F''(x) = f(x)$ and $G''(x) = g(x)$. It is now easy to check that (5.2) is the general solution of Eq. (5.1).

Combining the results of [2, Corollary 6.4, 1, Corollary 11.4, 18, Theorem 5.3] we obtain the following characterization of Darboux integrable equations:

THEOREM 5.3. *Equation* (3.1) *is Darboux integrable at level* $k \geqslant 0$ *if and only if* $H_p = K_q = 0$, *for some* $p, q \leqslant k$.

By Theorem 3.2 all hyperbolic equations (3.1) Darboux integrable at level zero are contact equivalent to the wave equation. Of course, this is a classical result due to Lie. Goursat [13] classified all f-Gordon equations Darboux integrable at level 1 and provided a closed form general solution to every equation on his list. Later Vessiot [30] redrived and refined Goursat's classification by more systematic methods.

As an easy consequence of Theorems 2.4 and 3.1 one obtains the following

THEOREM 5.4. *Hyperbolic Monge–Ampère equation* (2.9) *Darboux integrable at level one is contact equivalent to an f-Gordon equation.*

This theorem immediately implies that the Goursat–Vessiot list is a complete classification of Monge–Ampère equations that are Darboux integrable at level one. It is not clear to the author at this point, whether either Goursat or Vessiot were aware of this fact.

We now classify all hyperbolic equations (3.1) equivalent to an Euler–Lagrange equation of some Lagrangian that are Darboux integrable at level one independently of the Goursat–Vessiot results.

THEOREM 5.5. *A second-order scalar hyperbolic partial differential equation in the plane* (3.1) *is equivalent to an Euler–Lagrange equation of a first order Lagrangian and Darboux integrable at level one if and only if it is contact equivalent to wave equation* $s = 0$, *or Liouville equation* $s = e^u$, *or to the linear equation* $s = 2u/(x+y)^2$.

Proof. It is easy to verify that all three equations above are variational and Darboux integrable at level one. We now prove the converse. Let $L(x, y, u, p, q)$ be a first order Lagrangian for the equation (3.1). As in the proof of Theorem 3.6 one easily sees that $H_0 = K_0$. Moreover, the equation is quasi-linear, i.e., of the form

$$Ar + 2Bs + Ct + E = 0,$$

where A, B, C, and E, are first order functions, and so it is of Monge–Ampère type. If $H_0 = K_0 = 0$, then by Theorem 3.2 the equation is contact equivalent to the wave equation. Assume now $H_0 \neq 0$. By Theorem 5.3, $H_1 = K_1 = 0$, and so by Theorem 5.4 the equation is contact equivalent to some f-Gordon equation. By Theorem 4.3(i) the statement now follows. ∎

6. EXAMPLES

As our first example we consider the hyperbolic Fermi–Ulam–Pasta (FPU) equation

$$t = \kappa(p)^2\, r, \qquad \kappa(p) \neq 0. \tag{6.1}$$

We find $H_0 = K_0$ and one can also easily see that $L = -q^2/2 + \alpha(p)$, is a Lagrangian for (6.1), provided $\alpha''(p) = \kappa(p)$. Fackerell, Hartley, and Tucker [8] (see also [14]) showed that the equation (6.1) is Darboux integrable at level one if $\kappa(p)$ satisfies certain fourth-order differential equation. Recently Bryant, Griffiths, and Hsu [3, p. 92] proved that the FPU (6.1) is Darboux integrable at level one, if and only if it is equivalent to an FPU equation (6.1) with (a) $\kappa(p) = p^{-2}$, (b) $\kappa(p) = p^{-2/3}$, or $\kappa(p)$ given implicitly by either (c) $p = 2/\sqrt{\kappa(p)} + \ln((1 - \sqrt{\kappa(p)})/(1 + \sqrt{\kappa(p)}))$ or (d) $p = 2/\sqrt{\kappa(p)} + \ln((\sqrt{\kappa(p)} - 1)/(\sqrt{\kappa(p)} + 1)))$. In case (a) we find $H_0 = K_0 = 0$ and so the equation is contact equivalent to the wave equation. For the remaining cases we have $H_0 = K_0 \neq 0$ and $H_1 = 0$, and so by Theorem 4.3(i) the equations (b), (c), and (d), are equivalent to either the Liouville equation or the linear equation $s = 2u/(x + y)^2$. In fact, we will see below that all three equations are contact equivalent to $s = 2u/(x + y)^2$.

PROPOSITION 6.1. *The FPU equation* (6.1) *is contact equivalent to an equation of the form*

$$s = c(x + y)\, u, \tag{6.2}$$

for some function $c = c(w)$.

Proof. The characteristic vector fields for Eq. (6.1) are

$$X = D_y + \kappa(p)\, D_x \qquad \text{and} \qquad X = D_y - \kappa(p)\, D_x.$$

Note that $I = q - K(p)$ is an X invariant and $J = q + K(p)$ is an Y invariant function; here $K'(p) = \kappa(p)$. By Theorem 3.5, Eq. (6.1) is equivalent to an equation of the type $s = c(x, y, u)$. The contact transformation

$$\bar{x} = I, \quad \bar{y} = J, \quad \bar{u} = u - xp - yq, \quad \bar{p} = \frac{x - \kappa(p)\, y}{2\kappa(p)}, \quad \text{and} \quad \bar{q} = -\frac{x + \kappa(p)\, y}{2\kappa(p)},$$

transforms Eq. (6.1) into

$$\bar{s} + f(\bar{x} - \bar{y})(\bar{p} - \bar{q}) = 0, \qquad \text{where} \qquad f(w) = \frac{\kappa'(-w/2)}{2\kappa(-w/2)^2}. \tag{6.3}$$

The point transformation $x = \bar{x}$, $y = -\bar{y}$, and $u = \bar{u} \exp(-F(\bar{x} - \bar{y}))$, where $F'(w) = f(w)$, carries Eq. (6.3) into (6.2) with $c(w) = f(w)^2 - f'(w)$. ∎

Thus, examining the class of Eqs. (6.2) is the same as examining the class of FPU equations. The class of equations (6.2) has been extensively studied in the classical literature (see for instance [9, 11, 12]). For example, the Klein–Gordon equation or the whole class of variational Euler–Poisson–Darboux (EPD) equations belongs to equations of type (6.2). Putting a Darboux integrable FPU equation into the form (6.2) substantially simplifies the actual integration since we may use the method of Laplace (see [9, Chap. XII] for a detailed account of this method).

From Proposition 6.1 and Theorem 4.3 we immediately deduce the following classification result:

PROPOSITION 6.2. *The hyperbolic FPU equation $t = \kappa(p)^2 r$ satisfying $H_0 \neq 0$ and $H_1 = 0$ is contact equivalent to the linear equation $s = 2u/(x + y)^2$.*

We continue by investigating hyperbolic FPU equations Darboux integrable at higher levels. Preliminary work on the problem of classifying FPU equations satisfying $H_2 = K_2 = 0$ was done by Fackerell, Hartley, and Tucker [8] who derived, by brute force of computer algebra, a sixth-order differential equation for $\kappa(p)$ that has to be satisfied in order that (6.1) be Darboux integrable at level two. It is quite easy to compute H_2 for Eq. (6.1), and so we offer the following improvement of their result:

PROPOSITION 6.3. *The hyperbolic FPU equation $t = \kappa(p)^2 r$ satisfying $H_0 \neq 0$, $H_1 \neq 0$, and $H_2 = 0$, is contact equivalent to a linear equation of type $s = c(x + y)\, u$, where $c = c(w)$ satisfies the fourth-order differential equation*

$$\begin{aligned}(c^3 + (c')^2 - cc'')\, c^{(4)} + c(c''')^2 - 2c'(3c^2 + c'')\, c''' + (c'')^3 + 3c^2(c'')^2 \\ + c(7(c')^2 - 5c^3)\, c'' + 5(c^3 - (c')^2)(c')^2 + c^6 = 0.\end{aligned} \tag{6.4}$$

For equation $s = c(x+y)\,u$, we have

$$H_0 = K_0 = c(x+y)$$

and

$$H_1 = K_1 = \frac{c(x+y)^3 + (c'(x+y))^2 - c(x+y)\,c''(x+y)}{c(x+y)^2}.$$

More generally, using the recursion formulas (3.3) we deduce that $H_n = K_n = f_n(c^{[2n]}(x+y))$, for some function f_n, where $c^{[k]} = (c, c', c'', ..., c^{(k)})$.

PROPOSITION 6.4. *The hyperbolic FPU equation $t = \kappa(p)^2\, r$ satisfying $H_0 \neq 0$, $H_1 \neq 0$, ..., $H_{n-1} \neq 0$ and $H_n = 0$, is contact equivalent to a linear equation of type $s = c(x+y)\,u$, where $c = c(w)$ satisfies the $2n$th-order differential equation $f_n(c^{[2n]}(w)) = 0$.*

We note that Eq. (6.4) is $f_2(c^{[4]})\, f_1(c^{[2]}) = 0$.

It is a well-known result that the maximal connected group of contact transformations that preserve linear equations

$$s + a(x, y)\,p + b(x, y)\,q + c(x, y)\,u = 0$$

is the group of invertible transformations $\bar{x} = \psi(x)$, $\bar{y} = \xi(y)$, and $\bar{u} = \lambda(x, y)\,u$, where $\psi(x)$, $\xi(y)$ and $\lambda(x, y)$ are arbitrary functions. Using this fact it is not difficult to prove

PROPOSITION 6.5. *The maximal connected group of contact transformations that preserve the type of equations $s = c(x+y)\,u$ is a four-parameter group of point transformations*

$$\bar{x} = a_1 x + a_2, \qquad \bar{y} = a_1 y + a_3, \qquad \text{and} \qquad \bar{u} = a_4 u, \tag{6.5}$$

where a_1, a_2, a_3, and a_4 are real constants and $a_1 a_4 \neq 0$.

Let $c(w)$ and $\bar{c}(\bar{w})$ be two functions. If $s = c(x+y)\,u$ and $\bar{s} = \bar{c}(\bar{x}+\bar{y})\,\bar{u}$ are two equations that are transformed one to another by the point transformation (6.5), then $\bar{w} = aw + b$ and $\bar{c} = c/a^2$, where $a = a_1$ and $b = a_2 + a_3$. Prolonging this group action to a higher order jet space, we obtain $\bar{c}' = c'/a^3$, $\bar{c}'' = c''/a^4$, ..., $\bar{c}^{(n)} = c^{(n)}/a^{n+2}$, ... We have $\bar{f}_0(\bar{c}) = \bar{c} = c/a^2 = f_0(c)/a^2$. From the recursion formulas (3.3) and the induction argument easily follows that $\bar{f}_n(\bar{c}^{[2n]}) = f_n(c^{[2n]})/a^2$. Thus, if $c(w)$ is a solution to to the differential equation $f_n(c^{[2n]}) = 0$, then also $a^2 c(aw+b)$ is a solution. Note that the infinitesimal generators of this group action are $v_1 = \partial_w$ and $v_2 = w\partial_w - 2c\partial_c$. Using Lie reduction, we may reduce the order of the equation $f_n(c^{[2n]})=0$ by two, for $n\geqslant 1$. We have $[v_1, v_2]=v_1$. The algorithm for Lie reduction tells us to rewrite the equation in terms of the invariants $\hat{w}=c$ and $\hat{c}=c'$ of v_1. In these new variables the equation becomes $h_n(\hat{c}^{[2n-1]})=0$, for some function h_n. The reduced vector field $\hat{v}_2=2\hat{w}\partial_{\hat{w}}+3\hat{c}\partial_{\hat{c}}$ is a symmetry of $h_n(\hat{c}^{[2n-1]})=0$ and $\bar{w}=\sqrt{\hat{w}}/\hat{c}^{1/3}$ and $\bar{c}=\sqrt{\hat{w}}/\hat{c}'$ are invariants of v_2. In terms of the new variables equation $h_n(\hat{c}^{[2n-1]})=0$ reduces to a $(2n-2)^{\text{nd}}$-order equation $g(\bar{w}, \bar{c}^{[2n]})=0$. Hence, we expect, that there is a $(2n-2)$-parameter family of non-equivalent classes of FPU equations Darboux integrable at level n, for $n\geqslant 1$.

As our second example we consider equations that describe pseudo-spherical surfaces. A *pseudo-spherical surface* (p.p.s.) is a two-dimensional Riemannian manifold with constant Gaussian curvature -1. Consider a local orthonormal frame e_1, e_2 and its dual coframe ω^1, ω^2. If we denote ω^3 the connection form, then the following structure equations are satisfied

$$d\omega^1=\omega^2\wedge\omega^3, \qquad d\omega^2=\omega^1\wedge\omega^3, \qquad \text{and} \qquad d\omega^3=\omega^1\wedge\omega^2. \tag{6.6}$$

Following Chern and Tenenblat [7] we say that a scalar partial differential equation Δ for a function $u(x, y)$ *describes a p.s.s.* if Δ is the integrability condition for the existence of functions f_{ij}, $1\leqslant i\leqslant 3$, $1\leqslant j\leqslant 2$, depending on u and its derivatives such that the one-forms

$$\omega^1=f_{11}\,dx+f_{12}\,dy, \qquad \omega^2=f_{21}\,dx+f_{22}\,dy, \qquad \text{and} \qquad \omega^3=f_{31}\,dx+f_{32}\,dy,$$

satisfy the structure equations (6.6). Examples of such are KdV and mKdV equations, sin-Gordon equation, Klein–Gordon equation, Burgers equation, Liouville equation, etc. Let $\eta\in\mathbf{R}-S$, where S is the set of isolated points on the real line. We say that a differential equation Δ *describes an* η*-p.s.s.* if it describes a p.s.s. with $f_{21}=\eta$ and Δ is independent of η. Note that this definition is not invariant under the group of contact or point transformations.

For example, consider the forms

$$\omega^1=\omega^3=p\,dx+\frac{u}{\eta}\,dy \qquad \text{and} \qquad \omega^2=\eta\,dx+\frac{1}{\eta}\,dy,$$

which satisfy the structure equations (6.6) on the surface described by the equation $s=u$. Rabelo and Tenenblat [23] obtained that equations of type $s=F(u, p)$ that describe an η-p.s.s. are

$$s=F(u), \qquad \text{where} \qquad F''(u)+\alpha F(u)=0, \tag{6.7}$$

$$s=ve^{\delta u}\sqrt{\gamma p^2+\beta}, \tag{6.8}$$

$$s=ve^{\delta u}p, \tag{6.9}$$

and

$$s = \lambda u + \zeta p + \tau, \tag{6.10}$$

where α, β, γ, δ ν, λ, τ, ζ are real constants, with α, β, γ, δ ν non-zero. Equations (6.7) are

$$s = c_1 e^{au} + c_2 e^{-au}, \tag{6.11}$$

and

$$s = c_1 \cos(au) + c_2 \sin(au), \tag{6.12}$$

where a, c_1, c_2 are arbitrary real constants. We consider the four-parameter Lie group of transformations

$$\bar{x} = a_1 x, \qquad \bar{y} = a_2 y, \qquad \bar{u} = a_3 u + a_4, \tag{6.13}$$

where a_1, a_2, a_3, a_4, are real parameters. If $c_1 = c_2 = 0$, then we have a wave equation $s = 0$. If $c_1 = 0$ and $c_2 \neq 0$ or $c_2 = 0$ and $c_1 \neq 0$ we can use (6.13) to normalize Eqs. (6.11) to the Liouville equation $s = e^u$. If $c_1 \neq 0$ and $c_2 \neq 0$ we can normalize Eqs. (6.11) to either $s = \sinh u$ or $s = \cosh u$. If $c_1^2 + c_2^2 \neq 0$, we can normalize Eqs. (6.12) to $s = \sin u$. Note that if we allow the group parameters a_1, a_2, a_3, a_4, to be complex, then the equations $s = \sin u$, $s = \sinh u$, and $s = \cosh u$ are equivalent.

Next we consider Eqs. (6.8). An easy computation shows that $H_1 = K_1 = 0$ for all these equations and so they are Darboux integrable at level one. We first observe that β can be taken to be 1 or -1 without the loss of generality. Using the group (6.13), we transform Eqs. (6.8) into one of the four following equations $s = e^u \sqrt{\pm p^2 \pm 1}$. If we allow the group parameters to be complex then all these equations are equivalent to $s = e^u \sqrt{p^2 - 1}$, which is equation (C_I^2) on Vessiot's list [30].

Any equation of type (6.9) has $H_0 = 0$, $K_0 \neq 0$, and $K_1 = 0$. These equations can by normalized using (6.13) to $s = e^u p$, which interchanging x and y becomes equation (B_{II}) on the Vessiot's list.

Finally for Eqs. (6.10) we have $H_0 = K_0 = \lambda$, If $\lambda = 0$, then Eq. (6.10) is equivalent to $s = 0$. If $\lambda \neq 0$, then $H_1/H_0 = 1$ and so by Theorem 4.3(ii), Eq. (6.10) is equivalent $s = u$.

We have proved the following proposition.

PROPOSITION 6.6. *The equation $s = F(u, p)$, where F is a smooth function describes an η-p.s.s if and only if it is contact equivalent to one of the following equations*

(A1)	$s=u$,	(D1)	$s=0$,
(A2)	$s=\sin u$,	(D2)	$s=e^{u}p$,
(A3)	$s=\sinh u$,	(D3)	$s=e^{u}$,
(A4)	$s=\cosh u$,	(D4)	$s=e^{u}\sqrt{\pm p^{2}\pm 1}$.

If we allow complex parameters in the group (6.13), *Eqs.* (A2), (A3), *and* (A4), *are equivalent and Eqs.* (D4) *are equivalent.*

Note that equations (D1)–(D4) are all Darboux integrable at level one. Not all Darboux integrable equations of type $s=F(u,p)$ describe an η-p.s.s. As an example consider the equation $s=up$ which is Darboux integrable at level two ($H_0=0$ and $K_2=0$), but according to the last proposition, does not describe an η-p.s.s.

We now turn our attention to the nonlinear wave equation introduced by Calogero [5]

$$s+ur+f(p)=0, \qquad f(p)\neq 0, \tag{6.14}$$

which we also studied in [18, Chap. 6]. The characteristic vector fields are $X=uD_x+D_y$ and $Y=D_x$. We easily compute that $H_0=-r$, $K_0=r(f''(p)-2)$, and $H_1=0$. Since (6.14) is Monge–Ampère, by Theorem 5.3 of [18] there is an X invariant function I_1 of order one and an X invariant function I_2 of order two, namely, if $f(p)=1/g''(p)$, then $I_1=y+g'(p)$ and $I_2=rg''(p)\,e^{pg'(p)-g(p)}$. Obviously, $J_1=y$ is Y invariant function. Thus, Eq. (6.14) is contact equivalent to an f-Gordon equation. We also have $L=f(p)\,f'''(p)/(2-f''(p))$, and so for a generic function $f(p)$, the conditions of Theorem 3.4 are satisfied. Thus, (6.14) is contact equivalent to the equation of form (3.10). Indeed, the contact transformation

$$\bar{x}=I_1, \qquad \bar{y}=J_1, \qquad \bar{u}=px-u, \qquad \bar{p}=xf(p), \qquad \bar{q}=-q-xf(p),$$

transforms Eq. (6.14) into a linear equation

$$\bar{s}+a(\bar{x},\bar{y})\,\bar{p}+b(\bar{x},\bar{y})\,\bar{u}=0, \tag{6.15}$$

for some functions a and b. Since $H_1=0$, the last equation is integrable by the method of Laplace.

Now consider a special case of Eq. (6.14)

$$s+ur-2p(p+c)=0, \tag{6.16}$$

where c is a constant. We have $H_0 = -r$, $K_0 = -6$ and $H_1 = 0$. Therefore, $K_0/H_0 = 6$ and $H_1/H_0 = 0$ and so, by Theorem 4.2, Eq. (6.16) is contact equivalent to an EPD equation, namely Eq. (6.16) is equivalent to

$$s + \frac{p}{2x+y} + \frac{3q}{x+y} + \frac{3u}{2(x+y)^2} = 0.$$

Finally, we remark that Rabelo [22] proved that equations of type

$$s + ur + p^2 + b - a\sqrt{p^2+b} = 0, \tag{6.17}$$

where a, b, are real constants, describe an η-p.s.s. We have

$$K_0 = \frac{abr}{(p^2+b)^{3/2}}.$$

If $a=0$ or $b=0$, then $K_0 = 0$, otherwise $K_0 \neq 0$ and $K_1 = 0$. We conclude that every equation of the form (6.17) is contact equivalent to some linear equation of the form (6.15) Darboux integrable at evel one.

REFERENCES

1. I. M. Anderson and N. Kamran, "The Variational Bicomplex for Second-Order Scalar Partial Differential Equations in the Plane," CRM Technical Report, September 1994.
2. I. M. Anderson and N. Kamran, The variational bicomplex for second-order scalar partial differential equations in the plane, *Duke Math. J.* **87**, No. 2 (1997), 265–319.
3. R. Bryant, P. A. Griffiths, and L. Hsu, Hyperbolic exterior differential systems and their conservation laws, I, *Selecta Math. (N.S.)* **1** (1995), 22–112.
4. R. Bryant, P. A. Griffiths, and L. Hsu, Hyperbolic exterior differential systems and their conservation laws, II, *Selecta Math. (N.S.)* **1**, No. 2 (1995), 265–323.
5. F. Calogero, A solvable nonlinear equation, *Stud. Appl. Math.* **52** (1984), 189–199.
6. E. Cartan, Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, *Ann. Sci. École Norm. (3)* **27** (1910), 109–192.
7. S. S. Chern and K. Tenenblat, Pseudo-spherical surfaces and evolution equations, *Stud. Appl. Math.* **74** (1986), 55–83.
8. E. D. Fackerell, D. Hartley, and R. Tucker, An obstruction to the integrability of a class of nonlinear wave equations by 1-stable Cartan characteristics, *J. Differential Equations* **115** (1995), 153–165.
9. A. Forsyth, "Theory of Differential Equations," Vol. 6, Dover, New York, 1959.
10. R. B. Gardner and N. Kamran, Characteristics and the geometry of hyperbolic equations in the plane, *J. Differential Equations* **104** (1993), 60–117.
11. E. Goursat, "Leçon sur l'intégration des équations aux dériées partielles du second ordre à deux variables indépendantes," Tome 1, Hermann, Paris, 1896.
12. E. Goursat, "Leçon sur l'intégration des équations aux dérivées du second order à deux variables indépendantes," Tome 2, Hermann, Paris, 1896.
13. E. Goursat, Recherches sur quelques équations aux dérivées partielles du second ordre, *Ann. Fac. Toulouse (2)* **1** (1899), 31–78, 439–464.

14. D. Hartley, R. Tucker, and P. Tuckey, Equivalence of the Darboux and Gardner methods for integrating hyperbolic equations in the plane, *Duke Math. J.* **77**, No. 1 (1995), 167–192.
15. C. Hillyard, "A Maple Package for the Variational Calculus," Master's thesis, Utah State University, 1992.
16. N. H. Ibragimov, CRC handbook of Lie group analysis of differential equations, *in* "Symmetries, Exact Solutions and Conservation Laws," Vol. 1, CRC Press, Boca Raton, FL, 1995.
17. M. Juráš, "Geometric Aspects of Second-Order Scalar Partial Differential Equations in the Plane," Ph.D. thesis, Utah State University, 1997.
18. M. Juráš and I. M. Anderson, Generalized Laplace invariants and the method of Darboux, *Duke Math. J.* **89**, No. 2 (1997), 351–375.
19. S. Lie, Discussion der Differentialgleichung $s=f(z)$, *Arch. Math.* **6** (1880), 112–124.
20. A. V. Mikhailov, A. B. Shabat, and V. V. Sokolov, The symmetry approach to classification of integrable systems, *in* "What is integrability?" (V. E. Zakharov, Ed.), pp. 115–184, Springer-Verlag, New York, 1991.
21. L. V. Ovsiannikov, "Group Analysis of Differential Equations," Acadcemic Press, New York, 1982.
22. M. R. Rabelo, On equations which describe Pseudo-spherical surfaces, *Stud. Appl. Math.* **81** (1989), 221–248.
23. M. R. Rabelo and K. Tenenblat, On equations of type $u_{xt}=F(u, u_x)$ which describe pseudo-spherical surfaces, *J. Math. Phys.* **31** (1990), 1400–1407.
24. V. V. Sokolov and A. V. Zhiber, On the Darboux integrable hyperbolic equations, *Phys. Lett. A* **208** (1995), 303–308.
25. T. Tsujishita, Homological methods of computing invariants of systems of differential equations, *Differential Geom. Appl.* (1991), 3–34.
26. P. J. Vassiliou, Coupled systems of nonlinear wave equations and finite-dimensional Lie algebras, I, *Acta Appl. Math.* **8** (1987), 149–163.
27. P. J. Vassiliou, Coupled systems of nonlinear wave equations and finite-dimensional Lie algebras. II. A nonlinear system arising from the group $G_{6,1}$ and its exact integration, *Acta Appl. Math.* **8** (1987), 107–147.
28. P. J. Vassiliou, Geometry and the method of Laplace, *in* "Lie Theory, Differential Equations and Representation Theory" (V. Hussin, Ed.), pp. 395–404, Centre de Recherches Mathématiques, Montreal, 1989.
29. E. Vessiot, Sur les équations aux dérivées partielles du second order $F(x, y, z, p, q, r, s, t)=0$, intégrables par la méthode de Darboux, *J. Math. Pures Appl.* **18** (1939), 1–61.
30. E. Vessiot, Sur les équations aux dérivées partielles du second ordre $F(x, y, z, p, q, r, s, t)=0$, intégrables par la méthode de Darboux, *J. Math. Pures Appl.* **21** (1942), 1–66.